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Weak imposition of boundary conditions for the Navier–Stokes equations by a penalty method

Atife Caglar[1,*,†] and Anastasios Liakos[2]

[1] *Department of Mathematics, University of Wisconsin, Green Bay, WI 54302, U.S.A.*
[2] *Department of Mathematics, U.S. Naval Academy, Annapolis, MD 21402, U.S.A.*

SUMMARY

We prove convergence of the finite element method for the Navier–Stokes equations in which the no-slip condition and no-penetration condition on the flow boundary are imposed via a penalty method. This approach has been previously studied for the Stokes problem by Liakos (Weak imposition of boundary conditions in the Stokes problem. *Ph.D. Thesis*, University of Pittsburgh, 1999). Since, in most realistic applications, inertial effects dominate, it is crucial to extend the validity of the method to the nonlinear Navier–Stokes case. This report includes the analysis of this extension, as well as numerical results validating their analytical counterparts. Specifically, we show that optimal order of convergence can be achieved if the computational boundary follows the real flow boundary exactly. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The equilibrium flow of a viscous incompressible fluid is described by the stationary, incompressible Navier–Stokes equations. We consider these equations in a bounded polyhedral domain $\Omega \subset \mathbb{R}^d$, $d=2,3$, equipped with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -2Re^{-1}\nabla\cdot\mathbb{D}(\mathbf{u})+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f} &\quad \text{in } \Omega \\ \mathbf{u}=\mathbf{0} &\quad \text{on } \Gamma=\partial\Omega \\ \nabla\cdot\mathbf{u}=0 &\quad \text{in } \Omega \end{aligned} \tag{1}$$

*Correspondence to: Atife Caglar, Department of Mathematics, University of Wisconsin, Green Bay, WI 54302, U.S.A.
†E-mail: caglara@uwgb.edu

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Here, Re is the Reynolds number and $\mathbb{D}(\mathbf{u})$ is the deformation tensor given by

$$\mathbb{D}_{ij}(\mathbf{u})=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \quad \text{for } 1\leqslant i,\, j\leqslant d$$

The boundary Γ is assumed to be the union of k linear/planar segments Γ_j:

$$\Gamma=\bigcup_{j=1}^{k}\Gamma_j$$

Let $\mathbf{n}$ be the outward unit normal vector on Γ and $\boldsymbol{\tau}_i$, $i=1,\ldots,d-1$, a system of orthonormal tangential vectors.

The Navier–Stokes equations (1) occur as a limiting case of slightly compressible flows, small stresses and idealized boundaries. Difficulties can still occur while matching solutions of (1) to real fluid behavior. These typically occur in flows with large stresses and complex boundaries. The stabilized numerical methods have been proven useful in different stress contexts where the stabilization corresponds to a deviation from the idealized situation in the direction of an ignored physical process. Examples include nonlinear stress–strain relations, e.g. the Smagorinsky model, Smagorinsky [1], which Ladyzhenskaya [2] used as numerical regularization in under-resolved flow problems. To this end, this work deals with the imposition of no-slip condition $\mathbf{u}\cdot\boldsymbol{\tau}_i=0$ on Γ for the Navier–Stokes equations via a penalty method. Our interest in penalty methods, as opposed to the strong imposition of the no-slip boundary condition, stems from the following reasons:

- For under-resolved flows, weak imposition of boundary conditions, referred to as *weak boundary conditions*, seems to give more accurate solutions, c.f. Volker *et al.* [3]. Interestingly, Collis [4] has shown experiments where weak boundary conditions give smaller errors than strong boundary conditions.
- Weak boundary conditions are the standard approach in large eddy simulation and turbulence (near wall modeling/wall laws) with little mathematical support, which necessitates their study.
- In certain instances, there is uncertainty in the location of the boundary (e.g. roughness) or unknown boundary values (e.g. inflow conditions reconstructed from a few observations). In such cases, it makes sense to under weigh the boundary conditions.

The penalty method was first introduced by Courant [5]. He proposed a perturbed variational formulation for the approximate solution of elliptic problems with essential boundary conditions. Babuška [6] used a finite element version of Courant's method. Falk and King [7] used penalty methods to impose the incompressibility constraint, $\nabla\cdot\mathbf{u}=0$, weakly. Recent works where 'penalized' boundary conditions were studied include Carey and Krishnan [8], Layton [9] and Liakos [10].

We begin by decomposing the boundary condition $\mathbf{u}=\mathbf{0}$ to two separate conditions:

$$\text{'no-penetration'}: \mathbf{u}\cdot\mathbf{n}=0 \quad \text{on } \Gamma$$

$$\text{'no-slip'}: \mathbf{u}\cdot\boldsymbol{\tau}_i=0 \quad \text{on } \Gamma$$

Consequently, we impose these conditions as penalty terms or Lagrange multipliers.

In Section 2 we lay the mathematical foundation of the proposed method. We introduce all necessary spaces as well as examine the weak formulation of the Navier–Stokes equations with

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the no-penetration and no-slip conditions imposed via Lagrange multipliers. Section 3 includes the analysis of the continuous penalty–penalty method. Existence and uniqueness of solutions of our proposed method are exhibited, as well as bounds on the difference between the solutions of the weak formulation of the Navier–Stokes equations where the b.c. are imposed via Lagrange multiplier and penalty method, respectively. The former method is considered as our reference point since it arises naturally from the Galerkin formulation. In Section 4, we examine the discrete formulation of the Navier–Stokes equations where the b.c. are imposed via Lagrange multiplier. Next, in Section 5, the discrete penalty–penalty method is considered and the discrete analogue of error estimates of Section 3 is realized. Finally, we conclude our study with two numerical tests, which validate our theoretical estimates.

2. CONTINUOUS FORMULATION

We introduce the following spaces:

$$X=[H^1(\Omega)]^d$$

$$X_0=[H_0^1(\Omega)]^d$$

$$V=\{\mathbf{v}\in X_0 \mid \langle\nabla\cdot\mathbf{v},q\rangle_\Omega=0,\ \forall q\in L_0^2(\Omega)\}$$

$$V_0=\{\mathbf{v}\in V \mid \mathbf{v}\cdot\mathbf{n}=\mathbf{v}\cdot\boldsymbol{\tau}_1=\mathbf{v}\cdot\boldsymbol{\tau}_2=0 \text{ on } \Gamma\}$$

$$Y=L^2(\Omega)$$

$$Y_0=L_0^2(\Omega)=\{q\in L^2(\Omega) \mid \langle q,1\rangle_\Omega=0\}$$

$$Z=\prod_{j=1}^{k} H^{-1/2}(\Gamma_j)$$

The usual inner product in $L^2(\Omega)$ is denoted by $\langle .,.\rangle_\Omega$, with induced norm $\|\cdot\|$. $H^k(\Omega)$ is the $W^{k,2}(\Omega)$ Sobolev space with norm $\|.\|_{k,\Omega}$, and seminorm $|.|_{k,\Omega}$. The space $H^{-k}(\Omega)$ is the dual of $H_0^k(\Omega)$, which consists of functions in $H^k(\Omega)$ that vanish on Γ. The spaces $H^{k-1/2}(\Gamma)$ consist of the traces on Γ of all functions in $H^k(\Omega)$. Analogously, we denote by $H^{-(k-1/2)}(\Gamma)$ the dual space of $H^{k-1/2}(\Gamma)$ with $\langle .,.\rangle_\Gamma$ being the duality pairing, see Adams [11].

The norm $\|.\|_\Gamma$ of a function $\mathbf{u}\in\prod_{j=1}^{k} H^{1/2}(\Gamma_j)$ is defined by

$$\|\mathbf{u}\|_\Gamma=\left(\sum_{j=1}^{k}\|\mathbf{u}\|_{1/2,\Gamma_j}^2\right)^{1/2}$$

with dual norm $\|.\|_\Gamma^*:=\|.\|_Z$. In addition, we set $\|.\|_{\Gamma_j}=\|.\|_{H^{1/2}(\Gamma_j)}$ and $\|.\|_{\Gamma_j}^*=\|.\|_{H^{-1/2}(\Gamma_j)}$.

The most common weak formulation of the Navier–Stokes equations (1) is given by (see e.g. Girault and Raviart [12]).

Find $(\mathbf{u},p)\in(X_0,Y_0)$ *such that*:

$$\begin{aligned} a_0(\mathbf{u},\mathbf{v})+a_1(\mathbf{u};\mathbf{u},\mathbf{v})+a_2(\mathbf{v},p) &= \langle\mathbf{f},\mathbf{v}\rangle_\Omega \quad \forall\mathbf{v}\in X_0\\ a_2(\mathbf{u},q) &= 0 \quad \forall q\in Y_0 \end{aligned} \tag{2}$$

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where

$$a_0(\mathbf{u},\mathbf{v}) = 2Re^{-1}\int_\Omega \mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\,\mathrm{d}x$$

$$a_1(\mathbf{u};\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x$$

$$a_2(\mathbf{u},p) = -\int_\Omega p(\nabla\cdot\mathbf{u})\,\mathrm{d}x$$

An alternate form of the weak formulation 2 is obtained by considering the form

$$b(\mathbf{u};\mathbf{v},\mathbf{w}) = \tfrac{1}{2}[a_1(\mathbf{u};\mathbf{v},\mathbf{w}) - a_1(\mathbf{u};\mathbf{w},\mathbf{v})]$$

One may note that for $\mathbf{u}\in V_0$ and $\mathbf{v},\mathbf{w}\in X$, integration by parts leads to $b(\mathbf{u};\mathbf{v},\mathbf{w}) = \langle\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}\rangle_\Omega$. Thus $b(\mathbf{u};\mathbf{v},\mathbf{v})=0$. From this skew symmetry of the convective term and by the theory developed in Girault and Raviart [12] it is well known that an equivalent problem to (2) is

Find $\mathbf{u}\in V_0$ *such that*:

$$a(\mathbf{u};\mathbf{u},\mathbf{v}) = \langle\mathbf{f},\mathbf{v}\rangle_\Omega \quad \forall\mathbf{v}\in V_0 \tag{3}$$

where

$$a(\mathbf{u};\mathbf{v},\mathbf{w}) = a_0(\mathbf{u},\mathbf{w}) + b(\mathbf{u};\mathbf{v},\mathbf{w}) \tag{4}$$

Since the boundary conditions are imposed weakly, we must seek a formulation with velocities in X rather than X_0. Multiplying the first equation in (1) with a test function $\mathbf{v}\in X$ and using Green's formula, gives

$$a_0(\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + a_2(\mathbf{v},p) - \sum_{j=1}^{k}\langle\mathbb{S}(\mathbf{u},p)\mathbf{n}_j,\mathbf{v}\rangle_{\Gamma_j} = \langle\mathbf{f},\mathbf{v}\rangle_\Omega \quad \forall\mathbf{v}\in X \tag{5}$$

Here, the tensor $\mathbb{S}(.,.)$ is given by

$$\mathbb{S}_{ik}(\mathbf{u},p) = -p\delta_{ik} + 2Re^{-1}\mathbb{D}_{ik}(\mathbf{u}) + \tfrac{1}{2}\mathbf{u}_i\mathbf{u}_k \quad \text{for } 1\leqslant i,k\leqslant d$$

Since the pressure appears in the boundary integral, it turns out that we have to seek it in Y in all formulations, which are based on (5).

Next, we split the test function $\mathbf{v}$ in the boundary integral in (5) into its normal and tangential parts and define Lagrange multipliers on Γ_j by

$$\rho_j := -\mathbf{n}_j\cdot\mathbb{S}(\mathbf{u},p)\mathbf{n}_j \in H^{-1/2}(\Gamma_j)$$

$$\lambda_{1,j} := -\mathbf{n}_j\cdot\mathbb{S}(\mathbf{u},p)\boldsymbol{\tau}_{1,j} \in H^{-1/2}(\Gamma_j)$$

$$\lambda_{2,j} := -\mathbf{n}_j\cdot\mathbb{S}(\mathbf{u},p)\boldsymbol{\tau}_{2,j} \in H^{-1/2}(\Gamma_j)$$

Applying these substitutions on (5) yields:

$$a_0(\mathbf{u},\mathbf{v}) + b(\mathbf{u};\mathbf{u},\mathbf{v}) + a_2(\mathbf{v},p) + \sum_{j=1}^{k}[\langle\rho_j,\mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j} + \langle\lambda_{1,j},\mathbf{v}\cdot\boldsymbol{\tau}_{1,j}\rangle_{\Gamma_j} + \langle\lambda_{2,j},\mathbf{v}\cdot\boldsymbol{\tau}_{2,j}\rangle_{\Gamma_j}]$$
$$= \langle\mathbf{f},\mathbf{v}\rangle_\Omega \quad \forall\mathbf{v}\in X \tag{6}$$

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We define the multi-linear form

$$c(\mathbf{u}; p, \rho_j, \lambda_{1,j}, \lambda_{2,j}) := -a_2(\mathbf{u}, p) - \sum_{j=1}^{k} \langle \rho_j, \mathbf{u}\cdot\mathbf{n}_j \rangle_{\Gamma_j} - \sum_{i=1}^{2}\sum_{j=1}^{k} \langle \lambda_{i,j}, \mathbf{u}\cdot\boldsymbol{\tau}_{i,j} \rangle_{\Gamma_j}$$

and the space $\mathscr{K}$:

$$\mathscr{K} = \{\mathbf{u} \in X \mid c(\mathbf{u}; p, \rho, \lambda_1, \lambda_2) = 0,\ \forall p \in Y,\ \forall \rho, \lambda_1, \lambda_2 \in Z\}$$

which is defined in Liakos [10]. We are interested in solutions whose velocity $\mathbf{u}$, belongs to $\mathscr{K}$. Now the weak formulation is as follows:

Find $\mathbf{u} \in X$, $p \in Y$, $\rho, \lambda_1, \lambda_2 \in Z$ *such that*:

$$\begin{aligned} a(\mathbf{u}; \mathbf{u}, \mathbf{v}) - c(\mathbf{v}; p, \rho, \lambda_1, \lambda_2) &= \langle \mathbf{f}, \mathbf{v} \rangle_\Omega \\ c(\mathbf{u}; q, \sigma, \chi_1, \chi_2) &= 0 \end{aligned} \tag{7}$$

for all $(\mathbf{v}, q, \sigma, \chi_1, \chi_2) \in (X, Y, Z^3)$. Where $a(\cdot; \cdot, \cdot)$ is defined in (4). We associate (7) with the following problem:

Find $\mathbf{u} \in \mathscr{K}$ *such that*:

$$a(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega \quad \forall \mathbf{v} \in \mathscr{K} \tag{8}$$

Lemma 2.1

The space $\mathscr{K}$ is closed subspace of X. Further, $\mathscr{K} = V_0$.

Proof

The fact that $\mathscr{K}$ is closed follows from the continuity of the form $c(.,.,.,.,.)$, which is obtained by using the Cauchy–Schwartz and Hölder's inequality.

To show that $V_0 \subset \mathscr{K}$, let $\mathbf{v} \in V_0$ and let $q \in Y$ be arbitrary. Then there is a constant c such that $q + c \in Y_0$. Since $\langle \mathbf{v}, q + c \rangle_\Omega = 0$, straightforward computation gives $\langle \mathbf{v}, q \rangle_\Omega = 0$. The boundary terms in the definition of $\mathscr{K}$ vanish for all test functions since $\mathbf{v}$ vanishes on the boundary. Thus, $\mathbf{v} \in \mathscr{K}$.

Next, let $\mathbf{v} \in \mathscr{K}$. Choosing $\rho = \lambda_1 = \lambda_2 = 0$ gives $\langle \mathbf{v}, q \rangle_\Omega = 0$ for all $q \in Y$ and hence for all $q \in Y_0$. Choosing $q = \lambda_1 = \lambda_2 = 0$ gives

$$\sum_{j=1}^{k} \langle \rho_j, \mathbf{v}\cdot\mathbf{n}_j \rangle_{\Gamma_j} = 0 \quad \forall \rho \in Z$$

This holds especially for all $\rho \in L^2(\Gamma)$. Hence, $\mathbf{v}\cdot\mathbf{n}$ is orthogonal to $L^2(\Gamma)$, which means $\mathbf{v}\cdot\mathbf{n} = 0$ a.e. on Γ. In the same way, one obtains that $\mathbf{v}\cdot\boldsymbol{\tau}_1 = 0$ and $\mathbf{v}\cdot\boldsymbol{\tau}_2 = 0$ a.e. on Γ. Since $\{\mathbf{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ is an orthonormal system, it follows $\mathbf{v} = 0$ a.e. on Γ. From this, we get that $\mathbf{v} \in (H_0^1(\Omega))^d$, see Galdi [13]. $\square$

From the properties of V_0 we obtain coercivity of the bilinear form $a_0(\cdot, \cdot)$ in $\mathscr{K}$. In addition, the tri-linear form $b(\cdot; \cdot, \cdot)$ is bounded

$$N := \sup_{\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{K}} \frac{b(\mathbf{u}; \mathbf{v}, \mathbf{w})}{|\mathbf{u}|_1 |\mathbf{v}|_1 |\mathbf{w}|_1}$$

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Consequently, problems (3) and (8) are equivalent. Since the properties of problem (3) are well known (see [12]) we have the following corollary:

Corollary 2.1
Problem (8) has a solution $\mathbf{u}\in\mathscr{K}$. This solution is unique provided

$$\frac{1}{\alpha^2}N|\mathbf{f}|^*<1 \quad \text{where } |\mathbf{f}|^*:=\sup_{\mathbf{v}\in\mathscr{K}}\frac{\langle\mathbf{f},\mathbf{v}\rangle_\Omega}{|\mathbf{v}|_1}$$

For ease of notation, we define the following norm in X:

$$|||\mathbf{u}|||:=[\|\nabla\mathbf{u}\|^2+\|\mathbf{u}\cdot\mathbf{n}\|_\Gamma^2+\|\mathbf{u}\cdot\boldsymbol{\tau}_1\|_\Gamma^2+\|\mathbf{u}\cdot\boldsymbol{\tau}_2\|_\Gamma^2]^{1/2} \tag{9}$$

The reader may note that this norm is equivalent to $\|.\|_1$ on X: Decompose $X=\text{span}\{1\}\oplus\tilde{X}$. Then, the first term in (9) defines a norm in $\tilde{X}$ and the other terms define a norm in span$\{1\}$ such that $|||\cdot|||$ defines a norm in X.

In order to show that there exist a unique ρ,λ_1,λ_2, we need a proper inf–sup condition for the multi-linear form $c(\cdot;\cdot,\cdot,\cdot,\cdot)$.

Lemma 2.2
There is a constant $\beta>0$ such that

$$\inf_{\substack{\chi_1,\chi_2,\sigma\in Z\\ p\in Y}}\sup_{\mathbf{v}\in X}\frac{c(\mathbf{v};q,\sigma,\chi_1,\chi_2)}{|||\mathbf{v}|||[\|p\|^2+\|\chi_1\|_Z^2+\|\chi_2\|_Z^2+\|\sigma\|_Z^2]^{1/2}}\geqslant\beta \tag{10}$$

Proof
Detailed proof of this lemma and all other lemmata, which are stated in this work without proof, are available in Liakos [10]. (This lemma is also generalization of lemma in Verfürth [14].) □

3. THE CONTINUOUS PENALTY–PENALTY METHOD

The behavior of fluids near boundaries plays a key role in high Reynolds number flow. The transition from no slip at the boundary to the free stream velocity generates large amounts of vorticity (see e.g. [9]). It was observed by Serrin [15] (see also [16, 17]) that for high Reynolds number flow slip with friction boundary conditions are more appropriate than no-slip boundary condition. Hence, it makes sense to relax both the no-slip and no-penetration boundary conditions by imposing them as penalty terms. Note that the variational formulation of (1) with no-slip and no-penetration boundary conditions is the same as the variational formulation of (1) with slip with friction boundary conditions, see Caglar [18].

Let $\varepsilon_1,\varepsilon_2,\varepsilon_3$ be given penalty parameters. The continuous penalty–penalty method is defined as *Find* $\mathbf{u}_\varepsilon\in V$ *satisfying*:

$$a_0(\mathbf{u}_\varepsilon,\mathbf{v})+b(\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})+\sum_{i=1}^{2}\varepsilon_i^{-1}\sum_{j=1}^{k}\langle\mathbf{u}_\varepsilon\cdot\boldsymbol{\tau}_{i,j},\mathbf{v}\cdot\boldsymbol{\tau}_{i,j}\rangle_{\Gamma_j}+\varepsilon_3^{-1}\sum_{j=1}^{k}\langle\mathbf{u}_\varepsilon\cdot\mathbf{n}_j,\mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j}$$

$$=\langle\mathbf{f},\mathbf{v}\rangle\quad\forall\mathbf{v}\in V \tag{11}$$

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To simplify the presentation, we introduce the following variables:

$$\lambda_{1,j}^{\varepsilon} := \varepsilon_1^{-1}\mathbf{u}_{\varepsilon}\cdot\boldsymbol{\tau}_{1,j}$$

$$\lambda_{2,j}^{\varepsilon} := \varepsilon_2^{-1}\mathbf{u}_{\varepsilon}\cdot\boldsymbol{\tau}_{2,j}$$

$$\lambda_{3,j}^{\varepsilon} := \varepsilon_3^{-1}\mathbf{u}_{\varepsilon}\cdot\mathbf{n}_j$$

Then Equation (11) can be written as

$$\begin{aligned}
a_0(\mathbf{u}_\varepsilon,\mathbf{v})+b(\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})+\sum_{i=1}^{2}\sum_{j=1}^{k}\langle\lambda_{i,j}^{\varepsilon},\mathbf{v}\cdot\boldsymbol{\tau}_{i,j}\rangle_{\Gamma_j}&\\
+\sum_{j=1}^{k}\langle\lambda_{3,j}^{\varepsilon},\mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j} &= \langle\mathbf{f},\mathbf{v}\rangle \quad \forall\mathbf{v}\in V\\
\varepsilon_1^{-1}\sum_{j=1}^{k}\langle\mathbf{u}_\varepsilon\cdot\boldsymbol{\tau}_{1,j},\chi_1\rangle_{\Gamma_j} &= \sum_{j=1}^{k}\langle\lambda_1^{\varepsilon},\chi_1\rangle_{\Gamma_j} \quad \forall\chi_1\in Z\\
\varepsilon_2^{-1}\sum_{j=1}^{k}\langle\mathbf{u}_\varepsilon\cdot\boldsymbol{\tau}_{2,j},\chi_2\rangle_{\Gamma_j} &= \sum_{j=1}^{k}\langle\lambda_2^{\varepsilon},\chi_2\rangle_{\Gamma_j} \quad \forall\chi_2\in Z\\
\sum_{j=1}^{k}\langle\sigma,\mathbf{u}_\varepsilon\cdot\mathbf{n}_j\rangle_{\Gamma_j} &= 0 \quad \forall\sigma\in Z
\end{aligned} \tag{12}$$

We also define a third multi-linear form

$$d(.,.) : V\times Z^3\to\mathbb{R}$$

with norm

$$\|d\| := \sup_{0\neq\mathbf{v}\in V,\lambda^\varepsilon\in Z^3}\frac{d(\mathbf{v},\lambda^\varepsilon)}{\|\mathbf{v}\|\|\lambda^\varepsilon\|}$$

where

$$d(\mathbf{v},\lambda^\varepsilon) := \sum_{i=1}^{2}\sum_{j=1}^{k}\langle\lambda_i^{\varepsilon},\mathbf{v}\cdot\boldsymbol{\tau}_{i,j}\rangle_{\Gamma_j}+\sum_{j=1}^{k}\langle\rho_j^{\varepsilon},\mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j}$$

$$\lambda^\varepsilon = (\lambda_1^\varepsilon,\lambda_2^\varepsilon,\rho^\varepsilon)\in Z^3 \quad \text{and} \quad |||\mathbf{v}||| := [\|\mathbf{v}\cdot\boldsymbol{\tau}_1\|_\Gamma^2+\|\mathbf{v}\cdot\boldsymbol{\tau}_2\|_\Gamma^2+\|\mathbf{v}\cdot\boldsymbol{\tau}_3\|_\Gamma^2]$$

$$\|\lambda^\varepsilon\| = \left[\sum_{i=1}^{2}\|\lambda_i^\varepsilon\|_Z^2+\|\rho^\varepsilon\|_Z^2\right]^{1/2}$$

Now let K_d be the kernel of the multi-linear form $d(.,.)$ in V, i.e.:

$$K_d = \{\mathbf{u}_\varepsilon\in V : d(\mathbf{u},\lambda^\varepsilon)=0,\ \forall\lambda^\varepsilon\in Z^3\}$$

Lemma 3.1
The space K_d is a closed subspace of the Hilbert space X.

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Proof
The proof follows from continuity of $d(.,.)$. □

Now problem (12) takes the following form on K_d:

$$a(\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})=\langle\mathbf{f},\mathbf{v}\rangle \quad \forall\mathbf{v}\in K_d \tag{13}$$

By the abstract theory developed in Girault and Raviart [12] one can show existence and uniqueness of a solution of (13) in K_d by establishing the following:

(1) Coercivity: There exists an $\alpha\in\mathbb{R}$ such that

$$a(\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)\geqslant\alpha\|\mathbf{u}_\varepsilon\|_X \quad \text{for all } \mathbf{u}_\varepsilon\in K_d$$

(2) The space K_d is separable in X and, for all $\mathbf{v}\in K_d$, the mapping $\mathbf{u}_\varepsilon\rightarrow a(\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})$ is sequentially weakly continuous on K_d.

For coercivity one needs Korn's inequality in K_d, which is shown in the following lemma:

Lemma 3.2
Let $\mathbf{u}_\varepsilon\in K_d$. Then Korn's inequality

$$\|\nabla\mathbf{u}_\varepsilon\|\leqslant C_K(\Omega)\|\mathscr{D}(\mathbf{u}_\varepsilon)\|$$

and the Poincaré inequality

$$\|\mathbf{u}_\varepsilon\|\leqslant C_P(\Omega)\|\nabla\mathbf{u}_\varepsilon\|$$

hold.

Proof
Detailed proof of the lemma can be found in Caglar [19]. □

Corollary 3.1
The bilinear form $a_0(\cdot,\cdot)$ is coercive on K_d

$$\alpha\|\mathbf{u}_\varepsilon\|_1^2\leqslant a_0(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)$$

with $\alpha=2Re^{-1}\min\{C_K^{-2}(\Omega),C_K^{-2}(\Omega)C_P^{-2}(\Omega)\}$.

Also, K_d is separable since it is closed subset of the Hilbert space X (cf. Lemma 3.1). To complete our argument on existence we have left to show the following lemma:

Lemma 3.3
The multi-linear form $a(.,.,.)$ is weakly continuous.

Proof
Let $\mathbf{u}_\varepsilon$ be a function in K_d and let $\mathbf{u}_{\varepsilon m}$ be a sequence in K_d so that $\mathbf{u}_{\varepsilon m}\rightarrow\mathbf{u}_\varepsilon$ as $m\rightarrow\infty$. It is known by Rellich's theorem that $H^1(\Omega)$ can be embedded compactly in $L^2(\Omega)$. There exists a subsequence, still denoted $\mathbf{u}_{\varepsilon m}$ that converges strongly to $\mathbf{u}_\varepsilon$ in $L^2(\Omega)^d$ as $m\rightarrow\infty$. Let $\mathbf{v}$ be in a dense subset $L=\{\mathbf{w}\in D(\Omega)\subset V: c(\mathbf{w},\lambda^\varepsilon)=0,\ \text{for all } \lambda^\varepsilon\in Z^3\}$ of K_d. Note that $\mathbf{u}_{\varepsilon m}\in K_d$ and thus in V. Thus we have

$$\langle\mathbf{u}_{\varepsilon m}\cdot\nabla\mathbf{u}_{\varepsilon m},\mathbf{v}\rangle_\Omega=-\langle\mathbf{u}_{\varepsilon m}\cdot\nabla\mathbf{v},\mathbf{u}_{\varepsilon m}\rangle_\Omega$$

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Hence we may write $b(.;.,.)$ as

$$b(\mathbf{u}_{\varepsilon m};\mathbf{u}_{\varepsilon m},\mathbf{v})=\frac{1}{2}\langle\mathbf{u}_{\varepsilon m}\cdot\nabla\mathbf{u}_{\varepsilon m},\mathbf{v}\rangle_\Omega-\frac{1}{2}\langle\mathbf{u}_{\varepsilon m}\cdot\nabla\mathbf{v},\mathbf{u}_{\varepsilon m}\rangle_\Omega$$
$$=-\langle\mathbf{u}_{\varepsilon m}\cdot\nabla\mathbf{v},\mathbf{u}_{\varepsilon m}\rangle_\Omega-\sum_{i,j=1}^{d}\int_\Omega\mathbf{u}_{\varepsilon mi}\mathbf{u}_{\varepsilon mj}\mathbf{v}_{i,j}\,\mathrm{d}x$$

where $\mathbf{v}_{i,j}\in L^\infty(\Omega)$ (note $\mathbf{v}$ is infinitely many times differentiable) and $\lim_{m\to\infty}\mathbf{u}_{\varepsilon mi}\mathbf{u}_{\varepsilon mj}=\mathbf{u}_{\varepsilon i}\mathbf{u}_{\varepsilon j}\in L^1(\Omega)$. Hence

$$\lim_{m\to\infty}b(\mathbf{u}_{\varepsilon m};\mathbf{u}_{\varepsilon m},\mathbf{v})=-\sum_{i,j=1}^{d}\int_\Omega\mathbf{u}_{\varepsilon i}\mathbf{u}_{\varepsilon j}\mathbf{v}_{i,j}\,\mathrm{d}x$$
$$=-b(\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})=b(\mathbf{u}_\varepsilon;\nabla\mathbf{v},\mathbf{u}_\varepsilon)$$

since $\mathbf{u}_\varepsilon\in K_d\subset V$. Given that the form $a_0(.,.)$ is continuous and $\mathbf{u}_{\varepsilon m}\to\mathbf{u}_\varepsilon$ in $L^2(\Omega)$ implies that

$$\lim_{m\to\infty}a_0(\mathbf{u}_{\varepsilon m},\mathbf{v})=a_0(\mathbf{u}_\varepsilon,\mathbf{v})$$

Thus

$$\lim_{m\to\infty}a(\mathbf{u}_{\varepsilon m};\mathbf{u}_{\varepsilon m},\mathbf{v})=a(\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon,\mathbf{v})$$

for all $\mathbf{v}\in L$. By density of L in K_d and continuity of the forms $a(.;.,.)$ and $b(.;.,.)$ the result follows. □

Lemma 3.4
There is a constant $\beta''>0$ such that

$$\inf_{\sigma,\chi_1,\chi_2\in Z}\sup_{\mathbf{v}\in V}\frac{\sum_{j=1}^{k}(\langle\chi_1,\mathbf{v}\cdot\boldsymbol{\tau}_{1,j}\rangle_{\Gamma_j}+\langle\chi_2,\mathbf{v}\cdot\boldsymbol{\tau}_{2,j}\rangle_{\Gamma_j}+\langle\sigma,\mathbf{v}\cdot\mathbf{n}_j\rangle_{\Gamma_j})}{\|\mathbf{v}\|_1[\|\chi_1\|_Z^2+\|\chi_2\|_Z^2+\|\sigma\|_Z^2]^{1/2}}\geqslant\beta''$$

Proof
See Liakos [10]. □

Define

$$M:=\sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V}\frac{|b(\mathbf{u};\mathbf{v},\mathbf{w})|}{|\mathbf{u}|_1|\mathbf{v}|_1|\mathbf{w}|_1}$$

Proposition 3.1
Let $(\mathbf{u},\lambda_1,\lambda_2,\rho)$ be the solution of (7) and $(\mathbf{u}_\varepsilon,\lambda_1^\varepsilon,\lambda_2^\varepsilon,\lambda_3^\varepsilon)$ the solution to (12). Then

$$e_\varepsilon(\lambda_1,\lambda_2,\rho)+\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{1,\Omega}\leqslant C\left[\sum_{i=1}^{2}\varepsilon_i^2\|\lambda_i\|_\Gamma^2+\varepsilon_3^2\|\rho\|_\Gamma^2\right]^{1/2}\tag{14}$$

where

$$e_\varepsilon(\lambda_1,\lambda_2,\rho):=[\|\lambda_1-\lambda_1^\varepsilon\|_Z^2+\|\lambda_2-\lambda_2^\varepsilon\|_Z^2+\|\rho-\lambda_3^\varepsilon\|_Z^2]^{1/2}$$

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and

$$C := \left[\frac{2(Re^{-1}+\alpha)}{\beta''} + \frac{2(Re^{-1}+\alpha)}{\alpha\beta''(1-M\alpha^{-2}|\mathbf{f}|^*)} \right]$$

Proof
Subtracting (12) from (7) yields

$$\begin{aligned} &a_0(\mathbf{u}-\mathbf{u}_\varepsilon, \mathbf{v}) + b(\mathbf{u}-\mathbf{u}_\varepsilon; \mathbf{u}, \mathbf{v}) + b(\mathbf{u}_\varepsilon; \mathbf{u}-\mathbf{u}_\varepsilon, \mathbf{v}) \\ &\quad + \sum_{i=1}^{2} \sum_{j=1}^{k} \langle \lambda_i - \lambda_i^\varepsilon, \mathbf{v}\cdot\boldsymbol{\tau}_{i,j} \rangle_{\Gamma_j} + \sum_{j=1}^{k} \langle \rho - \lambda_3^\varepsilon, \mathbf{v}\cdot\mathbf{n}_j \rangle_{\Gamma_j} = 0 \quad \forall \mathbf{v} \in V \end{aligned}$$

$$\sum_{j=1}^{k} \langle (\mathbf{u}-\mathbf{u}_\varepsilon)\cdot\boldsymbol{\tau}_{i,j}, \chi_i \rangle_{\Gamma_j} = -\varepsilon_i \sum_{j=1}^{k} \langle \lambda_i^\varepsilon, \chi_i \rangle_{\Gamma_j} \quad \forall \chi_i \in Z, \quad i=1,2 \tag{15}$$

$$\sum_{j=1}^{k} \langle (\mathbf{u}-\mathbf{u}_\varepsilon)\cdot\mathbf{n}_j, \chi_3 \rangle_{\Gamma_j} = -\varepsilon_3 \sum_{j=1}^{k} \langle \lambda_3^\varepsilon, \chi_3 \rangle_{\Gamma_j} \quad \forall \chi_3 \in Z$$

From Equation (14) we have

$$e_\varepsilon(\lambda_1, \lambda_2, \rho) \leqslant \frac{2(Re^{-1}+\alpha)}{\beta''} \|\mathbf{u}-\mathbf{u}_\varepsilon\|_1 \tag{16}$$

Let $\mathbf{v} = \mathbf{u}-\mathbf{u}_\varepsilon$ in Equation (15) then

$$a_0(\mathbf{u}-\mathbf{u}_\varepsilon, \mathbf{u}-\mathbf{u}_\varepsilon) + b(\mathbf{u}-\mathbf{u}_\varepsilon; \mathbf{u}, \mathbf{u}-\mathbf{u}_\varepsilon) - \sum_{i=1}^{2} \varepsilon_i \sum_{j=1}^{k} \langle \lambda_i^\varepsilon, \lambda_i - \lambda_i^\varepsilon \rangle_{\Gamma_j} - \varepsilon_3 \sum_{j=1}^{k} \langle \lambda_3^\varepsilon, \rho - \lambda_3^\varepsilon \rangle_{\Gamma_j} = 0 \tag{17}$$

Thus

$$\begin{aligned} a_0(\mathbf{u}-\mathbf{u}_\varepsilon, \mathbf{u}-\mathbf{u}_\varepsilon) &= b(\mathbf{u}-\mathbf{u}_\varepsilon; \mathbf{u}_\varepsilon, \mathbf{u}-\mathbf{u}_\varepsilon) - \sum_{i=1}^{2} \varepsilon_i \sum_{j=1}^{k} \langle \lambda_i^\varepsilon, \lambda_i - \lambda_i^\varepsilon \rangle_{\Gamma_j} - \varepsilon_3 \sum_{j=1}^{k} \langle \lambda_3^\varepsilon, \rho - \lambda_3^\varepsilon \rangle_{\Gamma_j} \\ &\leqslant M\alpha^{-1}|f|^*|\mathbf{u}-\mathbf{u}_\varepsilon|_1^2 + \sum_{i=1}^{2} \varepsilon_i \sum_{j=1}^{k} \langle \lambda_i, \lambda_i - \lambda_i^\varepsilon \rangle_{\Gamma_j} + \varepsilon_3 \sum_{j=1}^{k} \langle \rho, \rho - \lambda_3^\varepsilon \rangle_{\Gamma_j} \\ &\leqslant \alpha|\mathbf{u}-\mathbf{u}_\varepsilon|_1^2 + \sum_{i=1}^{2} \varepsilon_i \|\lambda_i\|_\Gamma \|\lambda_i - \lambda_i^\varepsilon\|_Z + \|\rho\|_\Gamma \|\rho - \lambda_3^\varepsilon\|_Z \\ &\leqslant \alpha|\mathbf{u}-\mathbf{u}_\varepsilon|_1^2 + e_\varepsilon(\lambda_1, \lambda_2, \rho) \left[\sum_{i=1}^{2} \varepsilon_i^2 \|\lambda_i\|_\Gamma^2 + \varepsilon_3^2 \|\rho\|_\Gamma^2 \right]^{1/2} \end{aligned} \tag{18}$$

Combining (16) and (18), as well as coercivity of $a_0(.,.)$ yields the result. □

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4. FINITE ELEMENT SPACES

The polyhedral domain Ω is subdivided into d-simplices with sides of length less than h with $\mathscr{T}^h$ being the family of partitions. We will assume that $\mathscr{T}^h$ satisfies the usual regularity assumptions see e.g. Ciarlet [20] that

(1) Each vertex of Ω is a vertex of a simplex $T\in\mathscr{T}^h$.
(2) Each simplex $T\in\mathscr{T}^h$ has at least one vertex in the interior of Ω.
(3) Any two d-simplices $T, T'\in\mathscr{T}^h$ may meet in a vertex, a whole edge, or a whole face.
(4) Each simplex $T\in\mathscr{T}^h$ contains a ball with radius $c_0 h$ and is contained in a ball with radius $c_1 h$.

The constants c_0, c_1 denote different constants, which are independent of h. Denote by $\mathscr{O}_j^h$ the partition of Γ_j, which is induced by $\mathscr{T}^h$. Let $X^h\subset X$, $Y^h\subset Y$, $Z^h\subset Z$, and

$$X_0^h=\{\mathbf{u}^h\in X^h \mid \mathbf{u}^h=0 \text{ on } \Gamma\}$$

The spaces X^h and Y^h are assumed to satisfy the following properties:

(I) There is a constant $\tilde{\beta}>0$ independent of h for which

$$\inf_{0\neq p^h\in Y^h}\sup_{0\neq \mathbf{u}^h\in X_0^h}\frac{\int_\Omega p^h \operatorname{div}\mathbf{u}^h\,\mathrm{d}x}{\|p^h\|_{0,\Omega}\|\mathbf{u}^h\|_{1,\Omega}}\geqslant\tilde{\beta}$$

(II) $\inf_{p^h\in Y^h}\|p-p^h\|_{0,\Omega}\leqslant ch\|p\|_{1,\Omega}\ \forall p\in H^1(\Omega)$.
(III) There exists a continuous linear operator $\Pi^h: H^1(\Omega)^d\to X^h$ for which

$$\Pi^h(H_0^1(\Omega)^d)\subset X_0^h$$

$$\|\mathbf{u}-\Pi^h\mathbf{u}\|_{s,\Omega}\leqslant ch^{t-s}\|\mathbf{u}\|_{t,\Omega}\quad \forall \mathbf{u}\in H^t(\Omega)\quad \text{with } s=0,1 \text{ and } t=1,2$$

$$\|\mathbf{u}-\Pi^h\mathbf{u}\|_{0,\Gamma}\leqslant ch^{1/2}\|\mathbf{u}\|_{1,\Omega}$$

where $\|.\|_{0,\Gamma}=(\sum_{j=1}^k\|.\|_{0,\Gamma_j})^{1/2}$.

Assumption (I) balances the influence of the constraint $\operatorname{div}\mathbf{u}=0$ and also implies that the spaces

$$V_0^h=\{\mathbf{v}^h\in X_0^h \mid \langle q^h,\nabla\cdot\mathbf{v}^h\rangle=0,\ \forall q^h\in Y^h\}$$

$$V^h=\{\mathbf{v}^h\in X^h \mid \langle q^h,\nabla\cdot\mathbf{v}^h\rangle=0,\ \forall q^h\in Y^h\}\supset V_0^h$$

are not empty. As usual V^h is not a subset of V and in particular, the functions of V^h are not divergence free. For the skew-symmetric form $b(.;.,.,.)$ we introduce

$$N^h=\sup_{\mathbf{u}^h,\mathbf{v}^h,\mathbf{w}^h\in X^h}\frac{b(\mathbf{u}^h,\mathbf{v}^h,\mathbf{w}^h)}{|\mathbf{u}^h|_1|\mathbf{v}^h|_1|\mathbf{w}^h|_1}$$

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and

$$|\mathbf{f}|^* = \sup_{\mathbf{v}^h \in V^h} \frac{\langle \mathbf{f}, \mathbf{v}^h \rangle_\Omega}{|\mathbf{v}^h|_1}$$

With the above notation, the discrete analogue of problem (7) is

Find $\mathbf{u}^h \in X^h$, $p^h \in Y^h$, $\rho^h, \lambda_1^h, \lambda_2^h \in Z^h$ *such that*:

$$\begin{aligned} a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) - c(\mathbf{v}^h;\ p^h, \rho^h, \lambda_1^h, \lambda_2^h) &= \langle \mathbf{f}, \mathbf{v}^h \rangle_\Omega \\ c(\mathbf{u}^h;\ q^h, \sigma^h, \chi_1^h, \chi_2^h) &= 0 \end{aligned} \tag{19}$$

for all $\mathbf{v}^h \in X^h, q^h \in Y^h, \sigma^h, \chi_1^h, \chi_2^h \in Z^h$.

Under Assumption (I) this is equivalent to the following problem in V^h:

Find $\mathbf{u}^h \in V^h$ *such that*:

$$a(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) = \langle \mathbf{f}, \mathbf{v}^h \rangle_\Omega$$

By the abstract theory developed in Girault and Raviart [21, p. 108], the discrete problem will have unique solution provided

$$\alpha^{-2} N^h |\mathbf{f}|^* \leqslant \alpha^* < 1$$

and assuming there is a constant $\widehat{\beta} > 0$, independent of h, such that

$$\inf_{\substack{\rho^h, \lambda_1^h, \lambda_2^h \in Z^h \\ p^h \in Y^h}} \sup_{\mathbf{v}^h \in X^h} \frac{c(\mathbf{v}^h; p^h, \rho^h, \lambda_1^h, \lambda_2^h)}{\|\mathbf{v}^h\|_1 [\|\lambda_1^h\|_Z^2 + \|\lambda_2^h\|_Z^2 + \|\rho^h\|_Z^2 + \|p^h\|_{0,\Omega}^2]^{1/2}} \geqslant \widehat{\beta} \tag{20}$$

The discrete inf–sup condition (20) will be satisfied if the appropriate finite element spaces X^h, Y^h, Z^h are chosen. These choices are given in Layton [9]. The following lemmata are also needed for our analysis:

Lemma 4.1

There exists a constant $\overline{\beta} > 0$ independent of h, such that

$$\inf_{\substack{\rho^h \in Z^h \\ \lambda_1^h, \lambda_2^h \in Z^h}} \sup_{\mathbf{v}^h \in X^h} \frac{\sum_{j=1}^k (\langle \lambda_1^h, \mathbf{v}^h \cdot \boldsymbol{\tau}_1^{(j)} \rangle_{\Gamma_j} + \langle \lambda_2^h, \mathbf{v}^h \cdot \boldsymbol{\tau}_2^{(j)} \rangle_{\Gamma_j} + \langle \rho^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j})}{\|\mathbf{v}^h\|_1 [\|\lambda_1^h\|_Z^2 + \|\lambda_2^h\|_Z^2 + \|\rho^h\|_Z^2]^{1/2}} \geqslant \overline{\beta}$$

Proof

See Liakos [10]. □

Lemma 4.2

There is a constant $\widehat{\beta} > 0$, independent of h such that

$$\inf_{\substack{\rho^h, \lambda_1^h, \lambda_2^h \in Z^h \\ p^h \in Y^h}} \sup_{\mathbf{v}^h \in X^h} \frac{c(\mathbf{v}^h; p^h, \rho^h, \lambda_1^h, \lambda_2^h)}{\|\mathbf{v}^h\|_1 [\|\lambda_1^h\|_Z^2 + \|\lambda_2^h\|_Z^2 + \|\rho^h\|_Z^2 + \|p^h\|_{0,\Omega}^2]^{1/2}} \geqslant \widehat{\beta}$$

Proof

See Liakos [10]. □

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5. THE DISCRETE PENALTY–PENALTY METHOD

Using the above notation we can write the discrete analogue of problem (11) as

Find $\mathbf{u}_\varepsilon^h \in X^h$ *and* $p_\varepsilon^h \in Y^h$ *satisfying*:

$$a_0(\mathbf{u}_\varepsilon^h, \mathbf{v}^h) + b(\mathbf{u}_\varepsilon^h; \mathbf{u}_\varepsilon^h, \mathbf{v}^h) + \sum_{i=1}^{2} \varepsilon_i^{-1} \sum_{j=1}^{k} \langle \mathbf{u}_\varepsilon^h \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j}$$

$$+ \varepsilon_3^{-1} \sum_{j=1}^{k} \langle \mathbf{u}_\varepsilon^h \cdot \mathbf{n}_j, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} - \langle p_\varepsilon^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega = \langle \mathbf{f}, \mathbf{v}^h \rangle \tag{21}$$

$$\langle q^h, \nabla \cdot \mathbf{u}_\varepsilon^h \rangle_\Omega = 0$$

for all $\mathbf{v}^h \in X^h$ *and* $q^h \in Y^h$. This is equivalent to finding $\mathbf{u}_\varepsilon^h \in V^h$ such that

$$a_0(\mathbf{u}_\varepsilon^h, \mathbf{v}^h) + b(\mathbf{u}_\varepsilon^h; \mathbf{u}_\varepsilon^h, \mathbf{v}^h) + \sum_{i=1}^{2} \varepsilon_i^{-1} \sum_{j=1}^{k} \langle \mathbf{u}_\varepsilon^h \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j}$$

$$+ \varepsilon_3^{-1} \sum_{j=1}^{k} \langle \mathbf{u}_\varepsilon^h \cdot \mathbf{n}_j, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = \langle \mathbf{f}, \mathbf{v}^h \rangle \tag{22}$$

for all $\mathbf{v}^h \in V^h$.

In this section we obtain optimal error estimates for the velocities in the H^1-norm, for the pressure p in the L^2-norm and for λ_1, λ_2, ρ in the $H^{-1/2}$-norm assuming the computational boundary follows the flow boundary exactly. This requires that the penalty parameter $\varepsilon = \varepsilon_1 = \varepsilon_2 = \varepsilon_3$ be scaled by h^k where k is the degree of approximating polynomial. Let $(\mathbf{u}, \lambda_1, \lambda_2, p, \rho)$ and $(\mathbf{u}^h, \lambda_1^h, \lambda_2^h, p^h, \rho^h)$ be solutions of (7) and (19), respectively. Subtracting (19) from (7) we get

$$a_0(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u}; \mathbf{u} - \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{u} - \mathbf{u}^h; \mathbf{u}^h, \mathbf{v}^h) - \langle p - p^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega$$

$$+ \sum_{i=1}^{2} \sum_{j=1}^{k} \langle (\mathbf{u} - \mathbf{u}^h) \cdot \boldsymbol{\tau}_i^{(j)}, \mathbf{v}^h \cdot \boldsymbol{\tau}_i^{(j)} \rangle_{\Gamma_j} + \sum_{j=1}^{k} \langle \rho - \rho^h, \mathbf{v}^h \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0 \tag{23}$$

for all $\mathbf{v}^h \in V^h$. We write

$$\mathbf{u} - \mathbf{u}^h = (\mathbf{u} - \mathbf{v}^h) - (\mathbf{u}^h - \mathbf{v}^h)$$

with $\mathbf{v}^h$ being the best approximation of $\mathbf{u}$ in V^h. In particular we take $\mathbf{v}^h \in V_0^h$. Let

$$\eta := \mathbf{u} - \mathbf{v}^h$$

$$\phi^h := \mathbf{u}^h - \mathbf{v}^h$$

Since $\Gamma^h = \Gamma$, all error terms which come from approximating the boundary vanish, thus Equation (23) becomes

$$a_0(\eta - \phi^h, \mathbf{v}^h) + b(\mathbf{u}; \eta - \phi^h, \mathbf{v}^h) + b(\eta - \phi^h; \mathbf{u}^h, \mathbf{v}^h) - \langle p - p^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega = 0 \tag{24}$$

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for all $\mathbf{v}^h \in V^h$. Equation (24) can be rewritten as

$$a_0(\phi^h, \mathbf{v}^h) + b(\mathbf{u}; \phi^h, \mathbf{v}^h) + b(\phi^h; \mathbf{u}^h, \mathbf{v}^h)$$
$$= a_0(\eta, \mathbf{v}^h) + b(\mathbf{u}; \eta, \mathbf{v}^h) + b(\eta; \mathbf{u}^h, \mathbf{v}^h) + \langle p - p^h, \nabla \cdot \mathbf{v}^h \rangle_\Omega \tag{25}$$

Setting $\mathbf{v}^h = \phi^h$ in (25), and applying Cauchy–Schwartz inequality furnishes

$$2Re^{-1}\|\mathscr{D}(\phi^h)\|^2 \leqslant 2Re^{-1}\|\mathscr{D}(\eta)\|\|\mathscr{D}(\phi^h)\| + 2N^h|f|_h^*\alpha^{-1}|\phi^h|_1|\eta|_1$$
$$+ N^h|f|_h^*\alpha^{-1}|\phi^h|_1^2 + \|p - p^h\||\phi^h|_1$$

Employing Korn's inequality and the fact that $X^h \subset X$ we have

$$2Re^{-1}\|\mathscr{D}(\phi^h)\| \leqslant 2Re^{-1}\|\mathscr{D}(\eta)\| + 2\alpha\|\mathscr{D}(\eta)\| + \alpha\|\mathscr{D}(\phi^h)\| + \|p - p^h\|$$

Thus, we get

$$C_1\|\mathscr{D}(\phi^h)\| \leqslant C_2\|\mathscr{D}(\eta)\| + \|p - p^h\| \tag{26}$$

where C_1 and C_2 depend on Re, α, and $|\mathbf{f}|^*$. Adding and subtracting η on the left-hand side of (26) we get

$$\|\mathscr{D}(\mathbf{u} - \mathbf{u}^h)\| \leqslant (1 + C_1/C_2)\|\mathscr{D}(\mathbf{u} - \mathbf{v}^h)\| + 1/C_1\|p - p^h\|$$

Taking infima yields:

$$\|\mathscr{D}(\mathbf{u} - \mathbf{u}^h)\| \leqslant (1 + C_1/C_2) \inf_{0 \neq \mathbf{v}^h \in V^h} \|\mathscr{D}(\mathbf{u} - \mathbf{v}^h)\| + 1/C_1 \inf_{0 \neq p^h \in Y^h} \|p - p^h\| \tag{27}$$

From the inf–sup condition (20) and Equation (25) we conclude that

$$\tilde{\beta}\|p - p^h\| \leqslant C_3\|\mathbf{u} - \mathbf{u}^h\|_1$$

Under the approximation assumption

$$\inf_{\mathbf{v}^h \in V^h} |\mathbf{u} - \mathbf{v}^h|_1 + \left(\inf_{q^h \in Y^h} \|p - q^h\|_0^2 + \inf_{\rho^h \in Z^h} \|\rho - \rho^h\|_Z^2 \right)^{1/2} \leqslant Ch^k \max\{\|\mathbf{f}\|_{-1}, \|\mathbf{u}\|_2\}$$

we get

$$\|\mathbf{u} - \mathbf{u}^h\|_1 \leqslant Ch^k$$

To bound $\|\mathbf{u}^h - \mathbf{u}_\varepsilon^h\|_1$ subtract Equation (22) from (19) and set $\mathbf{v}^h = \mathbf{u}^h - \mathbf{u}_\varepsilon^h$

$$a_0(\mathbf{u}^h - \mathbf{u}_\varepsilon^h, \mathbf{u}^h - \mathbf{u}_\varepsilon^h) + b(\mathbf{u}^h - \mathbf{u}_\varepsilon^h; \mathbf{u}_\varepsilon^h, \mathbf{u}^h - \mathbf{u}_\varepsilon^h) + \sum_{i=1}^{2} \varepsilon_i \sum_{j=1}^{k} \langle \lambda_i^h, \lambda_i^h - \lambda_i^{\varepsilon,h} \rangle_{\Gamma_j}$$
$$+ \varepsilon_3 \sum_{j=1}^{k} \langle \rho^h - \lambda_3^{\varepsilon,h}, (\mathbf{u}^h - \mathbf{u}_\varepsilon^h) \cdot \mathbf{n}_j \rangle_{\Gamma_j} = 0 \tag{28}$$

From the inf–sup condition in Lemma 4.1 and Equation (21) we conclude that

$$\widehat{\beta} e_\varepsilon^h(\lambda_1^{\varepsilon,h}, \lambda_2^{\varepsilon,h}, \lambda_3^{\varepsilon,h}) \leqslant C_4\|\mathbf{u}^h - \mathbf{u}_\varepsilon^h\|_1 \tag{29}$$

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where

$$e_\varepsilon{}^h(\lambda_1^{\varepsilon,h},\lambda_2^{\varepsilon,h},\lambda_3^{\varepsilon,h}) := [\|\lambda_1^h-\lambda_1^{\varepsilon,h}\|_Z^2+\|\lambda_2^h-\lambda_2^{\varepsilon,h}\|_Z^2+\|\rho^h-\lambda_3^{\varepsilon,h}\|_Z^2]^{1/2}$$

Assuming $\lambda_i \in H^{1/2}$, adding and subtracting the quantities

$$\sum_{i=1}^{2}\varepsilon_i\sum_{j=1}^{k}\langle\lambda_i,\lambda_i-\lambda_i^{\varepsilon,h}\rangle_{\Gamma_j} \quad \text{and} \quad \sum_{j=1}^{k}\langle\rho,(\mathbf{u}^h-\mathbf{u}_\varepsilon^h)\cdot\mathbf{n}_j\rangle_{\Gamma_j}$$

in (28) gives

$$a_0(\mathbf{u}^h-\mathbf{u}_\varepsilon^h,\mathbf{u}^h-\mathbf{u}_\varepsilon^h)+b(\mathbf{u}^h-\mathbf{u}_\varepsilon^h;\mathbf{u}_\varepsilon^h,\mathbf{u}^h-\mathbf{u}_\varepsilon^h)$$
$$+\sum_{i=1}^{2}\varepsilon_i\sum_{j=1}^{k}\langle\lambda_i,\lambda_i-\lambda_i^{\varepsilon,h}\rangle_{\Gamma_j}+\sum_{j=1}^{k}\langle\rho-\lambda_3^{\varepsilon,h},(\mathbf{u}^h-\mathbf{u}_\varepsilon^h)\cdot\mathbf{n}_j\rangle_{\Gamma_j}=0 \tag{30}$$

By the same argument as in the proof of Proposition 3.1 we obtain

$$\|\mathbf{u}^h-\mathbf{u}_\varepsilon^h\|_1 \leqslant C_6[\varepsilon_1^2\|\lambda_1\|_\Gamma^2+\varepsilon_2^2\|\lambda_2\|_\Gamma^2+\varepsilon_3^2\|\rho\|_\Gamma^2]^{1/2} \tag{31}$$

where C_6 depends on Re, α, $\widehat{\beta}$, and $|\mathbf{f}|^*$. Combining (29) and (31) yields

$$\|\mathbf{u}^h-\mathbf{u}_\varepsilon{}^h\|_1 \leqslant C\varepsilon$$

Thus from (27) and (30)

$$\|\mathbf{u}-\mathbf{u}_\varepsilon{}^h\|_1 \leqslant Ch^k+C\varepsilon \tag{32}$$

where C depends on C_1 and $\|\lambda_i\|_\Gamma$ for $i=1,2$. Thus (32) indicates that the proper choice of ε in (32) is $\varepsilon=h^k$. To sum up we state the following theorem.

Theorem 5.1
Assume that the discrete spaces satisfy the inf–sup condition stated in Lemma 4.1. In addition, $\lambda_i \in H^{1/2}(\Gamma)$, $\Gamma=\Gamma^h$ and $\varepsilon=h^k$. Then the error in velocity of ((7) and) the discrete solution of the penalty–penalty method (22) satisfies

$$\|\mathbf{u}-\mathbf{u}_\varepsilon^h\|_1 \leqslant Ch^k$$

6. NUMERICAL STUDIES

We consider the two-dimensional discrete penalty–penalty method (19). The implementation of this method into a finite element code is described in John [22]. We present two examples. In the first example the solution is known in order to validate the analytical results. In the second example, we examine the effect of applying the no-slip condition weakly on the length of the recirculating vortices in flows past a backward facing step.

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Example 6.1 (*Navier–Stokes equations with a prescribed solution*)
We consider the Navier–Stokes equations (19) in the unit square $\Omega = (0,1)^2$ with the prescribed solution

$$u_1 = 2\pi \sin^3(\pi x)\sin(\pi y)\cos(\pi y)$$

$$u_2 = -3\pi \sin^2(\pi x)\cos(\pi x)\sin^2(\pi y)$$

$$p = \cos(\pi x) + \cos(\pi y)$$

The computations were carried out with viscosity $v^{-1} = 100$. The Navier–Stokes equations are discretized using inf–sup stable pairs of finite element spaces on a quadrilateral grid. Let $\hat{K}$ be the reference unit square $(-1,1)^2$ and K be an arbitrary mesh cell. The reference map is denoted by $F_K : \hat{K} \to K$. We define the local finite element spaces

$$Q_k(\hat{K}) := \left\{ \sum_{i,j=0}^{k} a_{ij} x_1^i x_2^j \right\}, \quad Q_k(K) := \{ p = \hat{p} \circ F_K^{-1} : \hat{p} \in Q_k(\hat{K}) \}$$

$$P_k(\hat{K}) := \left\{ \sum_{0 \leqslant i+j \leqslant k} a_{ij} x_1^i x_2^j \right\}, \quad P_k(K) := \{ p = \hat{p} \circ F_K^{-1} : \hat{p} \in P_k(\hat{K}) \}$$

The global finite element spaces are given by

$$Q_k := \{ v \in H^1(\Omega) : v|_K \in Q_k(K) \}, \quad k \geqslant 1$$

$$P_k^{\mathrm{disc}} := \{ v \in L^2(\Omega) : v|_K \in P_k(K) \}, \quad k \geqslant 1$$

The Q_2/P_1^{disc} finite element discretization is second-order accurate in the H^1-seminorm of the velocity. Alternately, the Q_3/P_2^{disc} finite element discretization possesses third-order accuracy. The viscosity v is chosen so that the finite element spaces do not need a stabilization of the convective term such as the streamline-diffusion stabilization. Accordingly, we use the standard Galerkin discretization. The initial grid (level 0) consists of four squares with edges of length 0.5. In our tests, we have chosen the penalty parameters for the tangential and the normal velocity to be the same, i.e. $\varepsilon_1 = \varepsilon_3 = \varepsilon$.

We present results for the error in the velocity with respect to the H^1-seminorm (Tables I and IV) and the L^2-norm (Tables II and V) as well as the error in pressure with respect to the L^2-norm (Tables III and VI). One can clearly see that $|\mathbf{u} - \mathbf{u}_\varepsilon^h|_1$ behaves asymptotically like $C(\varepsilon + h^k)$. This confirms our theoretical results. The results for the other norms suggest that

$$\|\mathbf{u} - \mathbf{u}_\varepsilon^h\|_0 \leqslant C(\varepsilon + h^k), \quad \|p - p_\varepsilon^h\|_0 \leqslant C(\varepsilon + h^{k+1})$$

for smooth solutions. For both finite element discretizations it is notable that there is always one step with a very large order of convergence before the order tends to the asymptotic.

Example 6.2 (*Flow past a backward facing step*)
We consider the backward facing step problem defined by Gartling [23]. The domain of the flow is the channel given in Figure 1. On the inflow boundary, $\{x = 0, 0 \leqslant y \leqslant 0.5\}$, the parabolic inflow

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Table I. Example 6.1, Q_2/P_1^{disc} finite element discretization, $|\mathbf{u}-\mathbf{u}_\varepsilon^h|_1$ and order of convergence.

Level	Strong no slip		Weak no slip					
			$\varepsilon_1=\varepsilon_3=h$		$\varepsilon_1=\varepsilon_3=h^2$		$\varepsilon_1=\varepsilon_3=h^3$	
2	7.443−1	3.53	1.636+1	0.04	3.167+0	2.17	8.397−1	3.57
3	1.802−1	2.0	1.499+1	0.13	7.226−1	2.13	1.802−1	2.22
4	3.950−2	2.19	5.984+0	1.32	3.950−2	4.19	3.950−2	2.19
5	9.233−3	2.10	2.820+0	1.09	9.235−3	2.10	9.233−3	2.10
6	2.264−3	2.03	1.408+0	1.00	2.264−3	2.03	2.264−3	2.03
7	5.632−4	2.01	7.041−1	1.00	5.633−4	2.01	5.632−4	2.01
8	1.406−4	2.00	3.521−1	1.00	1.406−4	2.00	1.406−4	2.00

Table II. Example 6.1, Q_2/P_1^{disc} finite element discretization, $\|\mathbf{u}-\mathbf{u}_\varepsilon^h\|_0$ and order of convergence.

Level	Strong no slip		Weak no slip					
			$\varepsilon_1=\varepsilon_3=h$		$\varepsilon_1=\varepsilon_3=h^2$		$\varepsilon_1=\varepsilon_3=h^3$	
2	1.381−2	5.51	1.659+0		2.979−1	1.77	4.031−2	4.15
3	1.622−3	3.09	1.549+0	0.10	7.072−2	2.07	1.623−3	4.63
4	1.850−4	3.13	6.029−1	1.36	1.908−4	8.53	1.851−4	3.13
5	2.208−5	3.07	2.854−1	1.08	2.428−5	2.97	2.208−5	3.07
6	2.723−6	3.02	1.424−1	1.00	3.646−6	2.74	2.724−6	3.02
7	3.393−7	3.00	7.113−2	1.00	6.889−7	2.40	3.393−7	3.00
8	4.237−8	3.00	3.554−2	1.00	1.554−7	2.15	4.238−8	3.00

Table III. Example 6.1, Q_2/P_1^{disc} finite element discretization, $\|p-p_\varepsilon^h\|_0$ and order of convergence.

Level	Strong no slip		Weak no slip					
			$\varepsilon_1=\varepsilon_3=h$		$\varepsilon_1=\varepsilon_3=h^2$		$\varepsilon_1=\varepsilon_3=h^3$	
2	1.988−2	5.94	3.195+0		6.722−1	1.67	9.040−2	3.98
3	1.846−3	3.43	2.777+0	2.02	1.597−1	2.07	1.846−3	5.61
4	3.850−4	2.26	1.329+0	1.06	3.870−4	8.69	3.850−4	2.26
5	9.144−5	2.07	6.377−1	1.06	9.191−5	2.07	9.144−5	2.07
6	2.255−5	2.02	3.206−1	0.99	2.267−5	2.02	2.255−5	2.02
7	5.619−6	2.00	1.605−1	1.00	5.648−6	2.00	5.619−6	2.00
8	1.404−6	2.00	8.025−2	1.00	1.411−6	2.00	1.404−6	2.00

profile $16y(0.5-y)$ is prescribed. On the outflow boundary, $\{x=30, -0.5\leqslant y\leqslant 0.5\}$, we apply outflow boundary conditions, i.e. $(2\nu\mathbb{D}\mathbf{u}-pI)\mathbf{n}=\mathbf{0}$. On all other boundaries, no-slip conditions are given.

We present results for the Reynolds numbers $Re=\nu^{-1}=200$ and 800. In the low Reynolds number case, there is only one recirculating vortex behind the step, whereas for the high

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Table IV. Example 6.1, Q_3/P_2^{disc} finite element discretization, $|\mathbf{u}-\mathbf{u}_\varepsilon^h|_1$ and order of convergence.

Level	Strong no slip		Weak no slip $\varepsilon_1=\varepsilon_3=h^{1.5}$		Weak no slip $\varepsilon_1=\varepsilon_3=h^3$		Weak no slip $\varepsilon_1=\varepsilon_3=h^4$	
2	6.156−2	3.96	9.313+0		3.558−1	3.35	6.156−2	4.39
3	7.245−3	3.09	2.805+0	1.73	7.243−3	5.62	7.242−3	3.09
4	8.806−4	3.04	9.936−1	1.50	8.808−4	3.04	8.806−4	3.04
5	1.083−4	3.02	3.515−1	1.50	1.083−4	3.02	1.083−4	3.03
6	1.345−5	3.01	4.095−4	9.75	1.346−5	3.01	1.345−5	3.01
7	1.679−6	3.00	1.447−4	1.50	1.679−6	3.00	1.679−6	3.00

Table V. Example 6.1, Q_3/P_2^{disc} finite element discretization, $\|\mathbf{u}-\mathbf{u}_\varepsilon^h\|_0$ and order of convergence.

Level	Strong no slip		Weak no slip $\varepsilon_1=\varepsilon_3=h^{1.5}$		Weak no slip $\varepsilon_1=\varepsilon_3=h^3$		Weak no slip $\varepsilon_1=\varepsilon_3=h^4$	
2	7.203−4	5.01	9.343−1		3.529−2	3.21	7.211−4	6.81
3	4.303−5	4.07	2.841−1	1.72	4.421−5	9.64	4.305−5	4.07
4	2.764−6	3.96	1.005−1	1.50	3.015−6	3.87	2.764−6	3.96
5	1.756−7	3.98	3.552−2	1.50	2.306−7	3.71	1.756−7	3.98
6	1.103−8	3.99	2.703−5	10.36	2.169−8	3.41	1.103−8	3.99
7	6.906−10	4.00	9.558−6	1.50	2.432−9	3.16	6.906−10	4.00

Table VI. Example 6.1, Q_3/P_2^{disc} finite element discretization, $\|p-p_\varepsilon^h\|_0$ and order of convergence.

Level	Strong no slip		Weak no slip $\varepsilon_1=\varepsilon_3=h^{1.5}$		Weak no slip $\varepsilon_1=\varepsilon_3=h^3$		Weak no slip $\varepsilon_1=\varepsilon_3=h^4$	
2	6.804−4	4.56	1.966+0		7.981−2	3.39	6.788−4	8.29
3	4.203−5	4.02	6.346−1	1.63	4.317−5	10.85	4.205−5	4.01
4	3.682−6	3.51	2.266−1	1.49	3.858−6	3.48	3.682−6	3.51
5	3.963−7	3.21	8.020−2	1.50	4.215−7	3.19	3.963−7	3.22
6	4.733−8	3.07	2.594−5	11.59	5.061−8	3.06	4.733−8	3.07
7	5.845−9	3.02	9.173−6	1.50	6.262−9	3.01	5.845−9	3.02

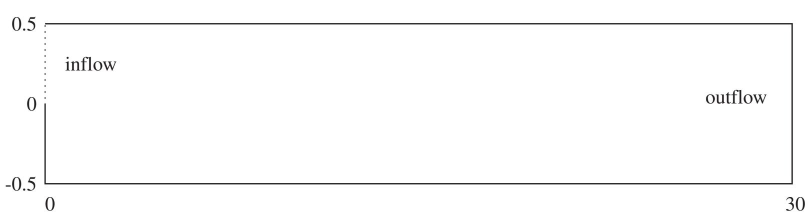

Figure 1. Domain for the backward facing step flow.

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Reynolds number case the flow possesses a second recirculating vortex on the upper wall, see Figures 2 and 3.

The Navier–Stokes equations (19) are discretized with the Q_2/P_1^{disc} finite element discretization. The initial grid (level 0) consists of 120 squares with edges of length 0.5. The choice of penalty parameters $\varepsilon_1 = \varepsilon_3 = h^2$ was used in the computations.

In the large Reynolds number case, we could not solve the discrete equations on coarse levels without stabilizing the convective term. On finer levels, one can see that the reattachment points of

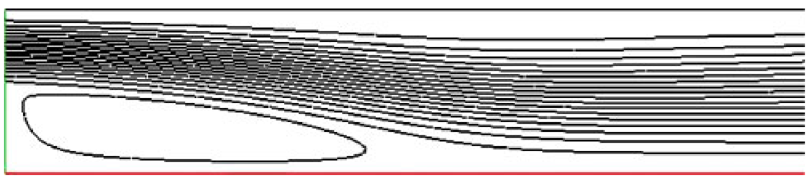

Figure 2. Streamlines of the velocity, backward facing step flow, $Re = 200$.

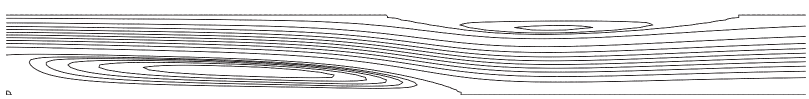

Figure 3. Streamlines of the velocity, backward facing step flow, $Re = 800$.

Table VII. Example 6.2, degrees of freedom.

Level	Velocity	Pressure	All
1	4338	1440	5778
2	16 354	5760	22 114
3	63 426	23 040	86 466
4	249 730	92 160	341 890
5	990 978	368 640	1 359 618
6	3 948 034	1 474 560	5 422 594

Table VIII. Example 6.2, $Re = 200$, reattachment point, penalty–penalty method.

Level	Reattachment
1	2.669353
2	2.765497
3	2.714913
4	2.668934
5	2.668990
6	2.669012

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Table IX. Example 6.2, $Re = 800$, reattachment and separation points, penalty–penalty method and reference values of Gartling [23].

Level	Reattachment lower wall	Separation upper wall	Reattachment upper wall
4	6.096100	4.852558	10.47939
5	6.096020	4.852671	10.47920
6	6.096108	4.852809	10.47920
Gartling [23]	6.10	4.85	10.48

both vortices as well as the separation point of the upper vortex, computed with the penalty–penalty method, are very close to the values given by Gartling [23], see Tables VII–IX.

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